

# On the Structure of the Minimum Critical Independent Set of a Graph

Vadim E. Levit

Ariel University Center of Samaria, Israel  
levitv@ariel.ac.il

Eugen Mandrescu

Holon Institute of Technology, Israel  
eugen\_m@hit.ac.il

## Abstract

Let  $G = (V, E)$ . A set  $S \subseteq V$  is *independent* if no two vertices from  $S$  are adjacent, and by  $\text{Ind}(G)$  we mean the set of all independent sets of  $G$ . The number  $d(X) = |X| - |N(X)|$  is the *difference* of  $X \subseteq V$ , and  $A \in \text{Ind}(G)$  is *critical* if

$$d(A) = \max\{d(I) : I \in \text{Ind}(G)\} \quad [7].$$

Let us recall the following definitions:

$$\ker(G) = \cap \{S : S \text{ is a critical independent set}\} \quad [5],$$

$$\text{core}(G) = \cap \{S : S \text{ is a maximum independent set}\} \quad [4].$$

Recently, it was established that  $\ker(G) \subseteq \text{core}(G)$  is true for every graph [5], while the corresponding equality holds for bipartite graphs [6].

In this paper we present various structural properties of  $\ker(G)$ . The main finding claims that

$$\ker(G) = \cup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.$$

**Keywords:** independent set, critical set,  $\ker$ , core, matching

## 1 Introduction

Throughout this paper  $G = (V, E)$  is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . If  $X \subseteq V$ , then  $G[X]$  is the subgraph of  $G$  spanned by  $X$ . By  $G - W$  we mean either the subgraph  $G[V - W]$ , if  $W \subseteq V(G)$ , or the partial subgraph  $H = (V, E - W)$  of  $G$ , for  $W \subseteq E(G)$ . In either case, we use  $G - w$ , whenever  $W = \{w\}$ .

The *neighborhood* of a vertex  $v \in V$  is the set  $N(v) = \{w : w \in V \text{ and } vw \in E\}$ , while the *closed neighborhood* of  $v \in V$  is  $N[v] = N(v) \cup \{v\}$ ; in order to avoid ambiguity,

we use also  $N_G(v)$  instead of  $N(v)$ . The *neighborhood* of  $A \subseteq V$  is denoted by  $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$ , and  $N[A] = N(A) \cup A$ .

A set  $S \subseteq V(G)$  is *independent* if no two vertices from  $S$  are adjacent, and by  $\text{Ind}(G)$  we mean the set of all the independent sets of  $G$ .

An independent set of maximum size will be referred to as a *maximum independent set* of  $G$ , and the *independence number* of  $G$  is  $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$ . Let  $\Omega(G)$  denote the family of all maximum independent sets, and  $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$  [4].

A *matching* is a set of non-incident edges of  $G$ ; a matching of maximum cardinality is a *maximum matching*, and its size is denoted by  $\mu(G)$ .

The number  $d(X) = |X| - |N(X)|$ ,  $X \subseteq V(G)$ , is called the *difference* of the set  $X$ . The number  $d_c(G) = \max\{d(X) : X \subseteq V\}$  is called the *critical difference* of  $G$ , and a set  $U \subseteq V(G)$  is *critical* if  $d(U) = d_c(G)$  [7]. The number  $id_c(G) = \max\{d(I) : I \in \text{Ind}(G)\}$  is called the *critical independence difference* of  $G$ . If  $A \subseteq V(G)$  is independent and  $d(A) = id_c(G)$ , then  $A$  is called *critical independent* [7]. Clearly,  $d_c(G) \geq id_c(G)$  is true for every graph  $G$ .

**Theorem 1.1** [7] *The equality  $d_c(G) = id_c(G)$  holds for every graph  $G$ .*

For a graph  $G$ , let denote  $\ker(G) = \cap\{S : S \text{ is a critical independent set}\}$ . It is known that  $\ker(G) \subseteq \text{core}(G)$  is true for every graph [5], while the equality holds for bipartite graphs [6].

For instance, the graph  $G$  from Figure 1 has  $X = \{v_1, v_2, v_3, v_4\}$  as a critical set, since  $N(X) = \{v_3, v_4, v_5\}$  and  $d(X) = 1 = d_c(G)$ , while  $I = \{v_1, v_2, v_3, v_6, v_7\}$  is a critical independent set, because  $d(I) = 1 = id_c(G)$ ; other critical sets are  $\{v_1, v_2\}$ ,  $\{v_1, v_2, v_3\}$ ,  $\{v_1, v_2, v_3, v_4, v_6, v_7\}$ . In addition,  $\ker(G) = \{v_1, v_2\}$ , and  $\text{core}(G)$  is a critical set.

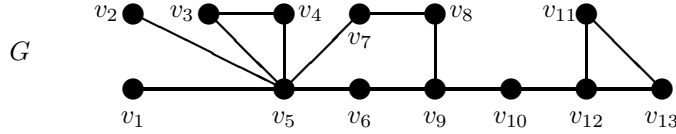


Figure 1:  $\text{core}(G) = \{v_1, v_2, v_6, v_{10}\}$ .

It is easy to see that all pendant vertices are included in every maximum critical independent set. It is known that the problem of finding a critical independent set is polynomially solvable [1, 7].

**Theorem 1.2** *For a graph  $G = (V, E)$ , the following assertions are true:*

(i) [5] *the function  $d$  is supermodular, i.e.,*

$$d(A \cup B) + d(A \cap B) \geq d(A) + d(B) \text{ for every } A, B \subseteq V;$$

(ii) [5]  *$G$  has a unique minimal critical independent set, namely,  $\ker(G)$ .*

(iii) [3] *there is a matching from  $N(S)$  into  $S$ , for every critical independent set  $S$ .*

In this paper we characterize  $\ker(G)$ . In addition, a number of properties of  $\ker(G)$  are presented as well.

## 2 Results

Deleting a vertex from a graph may decrease, leave unchanged or increase its critical difference. For instance,  $d_c(G - v_1) = d_c(G) - 1$ ,  $d_c(G - v_{13}) = d_c(G)$ , while  $d_c(G - v_3) = d_c(G) + 1$ , where  $G$  is depicted in Figure 1.

**Proposition 2.1** *Let  $G = (V, E)$  and  $v \in V$ . Then the following assertions hold:*

- (i)  $d_c(G - v) = d_c(G) - 1$  if and only if  $v \in \ker(G)$ ;
- (ii) if  $v \in \ker(G)$ , then  $\ker(G - v) \subseteq \ker(G) - \{v\}$ .

**Proof.** (i) Let  $v \in V$  and  $H = G - v$ .

If  $v \notin \ker(G)$ , then  $\ker(G) \subseteq V(G) - \{v\}$ . Hence

$$d_c(G - v) \geq |\ker(G)| - |N_H(\ker(G))| \geq |\ker(G)| - |N_G(\ker(G))| = d_c(G).$$

Consequently, we infer that  $d_c(G - v) < d_c(G)$  implies  $v \in \ker(G)$ .

Conversely, assume that  $v \in \ker(G)$ . Each  $u \in N(v)$  satisfies  $|N(u) \cap \ker(G)| \geq 2$ , because otherwise,  $d(\ker(G) - \{v\}) = d(\ker(G))$  and this contradicts the minimality of  $\ker(G)$ . Therefore,  $N(\ker(G) - \{v\}) = N(\ker(G))$  and hence

$$\begin{aligned} d(\ker(G) - \{v\}) &= |\ker(G) - \{v\}| - |N(\ker(G) - \{v\})| = \\ &= |\ker(G)| - 1 - |N(\ker(G))| = d_c(G) - 1. \end{aligned}$$

If there is some independent set  $A$  in  $G - v$ , such that  $d(A) = d_c(G)$ , then  $A$  is critical in  $G$  and, hence we get the following contradiction:  $v \in \ker(G) \subseteq A \subseteq V - \{v\}$ . Therefore,  $\ker(G) - \{v\}$  is a critical independent set of  $G - v$  and

$$d_c(G - v) = d(\ker(G) - \{v\}) = d_c(G) - 1.$$

(ii) Assume that  $\ker(G - v) \neq \emptyset$ . In part (i), we saw that  $\ker(G) - \{v\}$  is a critical independent set of  $G - v$ . Hence, we get that  $\ker(G - v) \subseteq \ker(G) - \{v\}$ . ■

**Remark 2.2** *Actually,  $\ker(G - v)$  may be different from  $\ker(G) - \{v\}$ ; for instance, if  $K_{3,2} = (A, B, E)$ ,  $|A| = 3$ , then  $\ker(K_{3,2}) = A$  and  $\ker(K_{3,2} - v) = \emptyset \neq \ker(K_{3,2}) - \{v\}$ , for every  $v \in A$ . It is also possible  $\ker(G) - \{v\} = \emptyset$ , while  $\ker(G - v) \neq \emptyset$ ; e.g.,  $G = C_4$ .*

By Theorem 1.2(iii), there is a matching from  $N(S)$  into  $S = \{v_1, v_2, v_3\}$ , for instance,  $M = \{v_2v_5, v_3v_4\}$ , since  $S$  is critical independent for the graph  $G$  from Figure 1. On the other hand, there is no matching from  $N(S)$  into  $S - v_3$ . The case of the critical independence set  $\ker(G)$  is more specific.

**Theorem 2.3** *Let  $A$  be a critical independent set in a graph  $G$ . Then the following statements are equivalent:*

- (i)  $A = \ker(G)$ ;
- (ii) there is no set  $B \subseteq N(A)$ ,  $B \neq \emptyset$  such that  $|N(B) \cap A| = |B|$ ;
- (iii) for each  $v \in A$  there exists a matching from  $N(A)$  into  $A - v$ .

**Proof.** (i)  $\implies$  (ii) By Theorem 1.2(iii), there is a matching, say  $M$ , from  $N(\ker(G))$  into  $\ker(G)$ . Suppose, to the contrary, that there is some non-empty set  $B \subseteq N(\ker(G))$  such that

$$|M(B)| = |N(B) \cap \ker(G)| = |B|.$$

It contradicts the fact that, by Theorem 1.2(ii),  $\ker(G)$  is a minimal critical independent set, because

$$d(\ker(G) - N(B)) = d(\ker(G)), \text{ while } \ker(G) - N(B) \subsetneq \ker(G).$$

(ii)  $\implies$  (i) Suppose  $A - \ker(G) \neq \emptyset$ . By Theorem 1.2(iii), there is a matching, say  $M$ , from  $N(A)$  into  $A$ . Since there are no edges connecting vertices belonging to  $\ker(G)$  with vertices from  $N(A) - N(\ker(G))$ , we obtain that  $M(N(A) - N(\ker(G))) \subseteq A - \ker(G)$ . Moreover, we have that  $|N(A) - N(\ker(G))| = |A - \ker(G)|$ , otherwise

$$\begin{aligned} |A| - |N(A)| &= (|\ker(G)| - |N(\ker(G))|) + (|A - \ker(G)| - |N(A) - N(\ker(G))|) > \\ &> (|\ker(G)| - |N(\ker(G))|) = d_c(G). \end{aligned}$$

It means that the set  $N(A) - N(\ker(G))$  contradicts the hypothesis of (ii), because

$$|N(A) - N(\ker(G))| = |A - \ker(G)| = |N(N(A) - N(\ker(G))) \cap A|.$$

Consequently, the assertion is true.

(ii)  $\implies$  (iii) By Theorem 1.2(iii), there is a matching, say  $M$ , from  $N(A)$  into  $A$ . Suppose, to the contrary, that there is no matching from  $N(A)$  into  $A - v$ . Hence, by Hall's Theorem, it implies the existence of a set  $B \subseteq N(A)$  such that  $|N(B) \cap A| = |B|$ , which contradicts the hypothesis of (ii).

(iii)  $\implies$  (ii) Assume, to the contrary, that there is a non-empty subset  $B$  of  $N(A)$  such that  $|N(B) \cap A| = |B|$ . Let  $v \in N(B) \cap A$ . Hence, we obtain that

$$|N(B) \cap A - v| < |B|.$$

Then, by Hall's Theorem, it is impossible to find a matching from  $N(A)$  into  $A - v$ , in contradiction with the hypothesis of (iii). ■

Since  $\ker(G)$  is a critical set, Theorem 1.2(iii) assures that there is a matching from  $N(\ker(G))$  into  $\ker(G)$ . The following result shows that there are at least two such matchings.

**Corollary 2.4** *For a graph  $G$  the following are true:*

- (i) *every edge  $e \in (\ker(G), N(\ker(G)))$  belongs to a matching from  $N(\ker(G))$  into  $\ker(G)$ ;*
- (ii) *every edge  $e \in (\ker(G), N(\ker(G)))$  is not included in one matching from  $N(\ker(G))$  into  $\ker(G)$  at least.*

**Proof.** Let  $e = xy \in (\ker(G), N(\ker(G)))$ , such that  $x \in \ker(G)$ . By Theorem 2.3(iii) there is a matching  $M$  from  $N(\ker(G))$  into  $\ker(G) - x$ , that matches  $y$  with some  $z \in \ker(G) - x$ . Clearly,  $M$  is a matching from  $N(\ker(G))$  into  $\ker(G)$  that does not contain the edge  $e = xy$ , while  $(M - \{yz\}) \cup \{xy\}$  is a matching from  $N(\ker(G))$  into  $\ker(G)$ , which includes the edge  $e = xy$ . ■

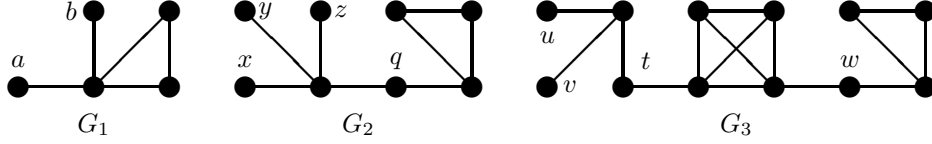


Figure 2:  $\text{core}(G_1) = \{a, b\}$ ,  $\text{core}(G_2) = \{q, x, y, z\}$ ,  $\text{core}(G_3) = \{t, u, v, w\}$ .

Let us notice that the graphs  $G_1$ ,  $G_2$  from Figure 2 have:  $\ker(G_1) = \text{core}(G_1)$ ,  $\ker(G_2) = \{x, y, z\} \subset \text{core}(G_2)$ , and both  $\text{core}(G_1)$  and  $\text{core}(G_2)$  are critical sets of maximum size. The graph  $G_3$  from Figure 2 has  $\ker(G_3) = \{u, v\}$ , the set  $\{t, u, v\}$  as a critical independent set of maximum size, while  $\text{core}(G_3) = \{t, u, v, w\}$  is not a critical set. If  $S_{\min}$  denotes an inclusion minimal independent set with  $d(S_{\min}) > 0$ , one can see that:  $S_{\min} = \ker(G_1)$  for  $G_1$ , while the graph  $G_2$  in the same figure has  $S_{\min} \in \{\{x, y\}, \{x, z\}, \{y, z\}\}$  and  $\ker(G_2) = \{x, y\} \cup \{x, z\} \cup \{y, z\}$ .

In [5] we have shown that  $\ker(G)$  is equal to the intersection of all critical, independent or not, sets of  $G$ .

**Theorem 2.5** *For every graph  $G$*

$$\ker(G) = \cup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}.$$

**Proof.** Let  $A$  be a critical set and  $S_0$  be an inclusion minimal independent set such that  $d(S_0) > 0$ . Then, Theorem 1.2(i) implies

$$d(A \cup S_0) + d(A \cap S_0) \geq d(A) + d(S_0) > d(A) = d_c(G).$$

Since  $S_0$  is an inclusion minimal independent set such that  $d(S_0) > 0$ , we obtain that if  $A \cap S_0 \neq S_0$ , then  $d(A \cap S_0) \leq 0$ . Hence

$$d(A) = d_c(G) \geq d(A \cup S_0) \geq d(A) + d(S_0) > d(A),$$

which is impossible. Therefore,  $S_0 \subseteq A$  for every critical set  $A$ . Consequently,

$$S_0 \subseteq \cap \{B : B \text{ is a critical set of } G\} = \ker(G).$$

Thus we obtain

$$\cup \{S_0 : S_0 \text{ is an inclusion minimal independent set such that } d(S_0) > 0\} \subseteq \ker(G).$$

Conversely, it is enough to show that every vertex from  $\ker(G)$  belongs to some inclusion minimal independent set with positive difference. Let  $v \in \ker(G)$ . According to Theorem 2.3(iii) there exists a matching, say  $M$ , from  $N(\ker(G))$  into  $\ker(G) - v$ .

Let us build the following sequence of sets

$$\{v\} \subseteq M(N(v)) \subseteq \dots \subseteq [MN]^k(v) \subseteq \dots,$$

where  $MN$  is a superposition of two mappings  $N : 2^V \rightarrow 2^V$  ( $N(A)$  is the neighborhood of  $A$ ) and  $M : 2^{N(\ker(G))} \rightarrow 2^{\ker(G)}$  ( $M(A)$  is set of the vertices matched by  $M$  with vertices belonging to  $A$ ).

Since the set  $\ker(G)$  is finite, there is an index  $j$  such that  $[MN]^j(v) = [MN]^{j+1}(v)$ . Hence  $|N([MN]^j(v))| = |[MN]^j(v)| - 1$ . In other words, we found an independent set, namely,  $[MN]^j(v)$  such that  $v \in [MN]^j(v)$  and  $d([MN]^j(v)) = 1$ . Therefore, there must exist an inclusion minimal independent set  $X$  such that  $v \in X$  and  $d(X) = 1$ . ■

**Remark 2.6** In a graph  $G$ , the union of all minimum cardinality independent sets  $S$  with  $d(S) > 0$  may be a proper subset of  $\ker(G)$ ; e.g., the graph  $G$  in Figure 3, that has  $\{x, y\} \subset \ker(G) = \{x, y, u, v, w\}$ .

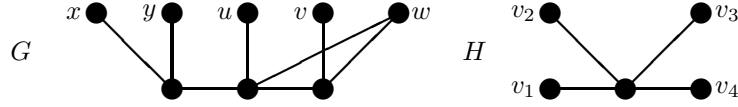


Figure 3: Both  $S_1 = \{x, y\}$  and  $S_2 = \{u, v, w\}$  are inclusion minimal independent sets satisfying  $d(S) > 0$ .

**Proposition 2.7**  $\min \{|S_0| : d(S_0) > 0, S_0 \in \text{Ind}(G)\} \leq |\ker(G)| - d_c(G) + 1$ .

**Proof.** Since  $\ker(G)$  is a critical independent set, Theorem 1.2(iii) implies that there is a matching, say  $M$ , from  $N(\ker(G))$  into  $\ker(G)$ . Let  $X = M(N(\ker(G)))$ . Then  $d(X) = 0$ . For every  $v \in \ker(G) - X$  we have

$$N(\ker(G)) \subseteq N(X) \subseteq N(X \cup \{v\}) \subseteq N(\ker(G)).$$

Hence we get  $|X \cup \{v\}| - |N(X \cup \{v\})| = 1$ , while  $|X \cup \{v\}| = |\ker(G)| - d_c(G) + 1$ . ■

**Remark 2.8** All the inclusion minimal independent sets  $S$ , with  $d(S) > 0$ , of the graph  $H$  from Figure 3 are of the same size. However, there are inclusion minimal independent sets  $S$  with  $d(S) > 0$ , of different cardinalities; e.g., the graph  $G$  from Figure 3.

**Proposition 2.9** If  $S_0$  is an inclusion minimal independent set with  $d(S_0) > 0$ , then  $d(S_0) = 1$ .

**Proof.** For each  $v \in S_0$ , it follows that  $N(S_0 - v) = N(S_0)$ , otherwise,

$$\begin{aligned} d(S_0 - v) &= |S_0 - v| - |N(S_0 - v)| = \\ &= |S_0| - 1 - |N(S_0 - v)| \geq |S_0| - |N(S_0)| > 0, \end{aligned}$$

i.e.,  $S_0$  is not an inclusion minimal independent set with positive difference.

Since  $S_0$  is an inclusion minimal independent set with positive difference, we know that  $d(S_0 - v) \leq 0$ . On the other hand, it follows from the equality  $N(S_0 - v) = N(S_0)$  that

$$d(S_0 - v) = |S_0 - v| - |N(S_0 - v)| = |S_0| - 1 - |N(S_0)| = d(S_0) - 1 \leq 0.$$

Consequently,  $0 < |S_0| - |N(S_0)| \leq 1$ , which means that  $|S_0| - |N(S_0)| = 1$ . ■

**Remark 2.10** *The converse of Proposition 2.9 is not true. For instance,  $S = \{x, y, u\}$  is independent in the graph  $G$  from Figure 3 and  $d(S) = 1$ , but  $S$  is not minimal with this property.*

**Proposition 2.11** *If  $S_i, i = 1, 2, \dots, k, k \geq 1$ , are inclusion minimal independent sets, such that  $d(S_i) > 0, S_i \not\subseteq \bigcup_{j=1, j \neq i}^k S_j, 1 \leq i \leq k$ , then  $d(S_1 \cup S_2 \cup \dots \cup S_k) \geq k$ .*

**Proof.** For  $k = 1$  the claim has been treated in Proposition 2.9, where we have achieved a stronger result.

We continue by induction on  $k$ .

Let  $k = 2$ . Since  $S_1 \neq S_1 \cap S_2 \subset S_1$ , it follows that  $d(S_1 \cap S_2) \leq 0$ . Hence, Theorem 1.2(i) and Proposition 2.9 imply

$$d(S_1 \cup S_2) \geq d(S_1 \cup S_2) + d(S_1 \cap S_2) \geq d(S_1) + d(S_2) = 2.$$

Assume that the assertion is true for each  $k \geq 2$ , and let  $\{S_i, 1 \leq i \leq k+1\}$  be a family of inclusion minimal independent sets with

$$d(S_i) > 0 \text{ and } S_i \not\subseteq \bigcup_{j=1, j \neq i}^{k+1} S_j, 1 \leq i \leq k+1.$$

Since  $S_{k+1} \neq (S_1 \cup S_2 \cup \dots \cup S_k) \cap S_{k+1} \subset S_{k+1}$ , we obtain that

$$d((S_1 \cup S_2 \cup \dots \cup S_k) \cap S_{k+1}) \leq 0.$$

Further, using the supermodularity of the function  $d$  and Proposition 2.9, we get

$$\begin{aligned} d(S_1 \cup S_2 \cup \dots \cup S_k \cup S_{k+1}) &\geq \\ &\geq d(S_1 \cup S_2 \cup \dots \cup S_k \cup S_{k+1}) + d((S_1 \cup S_2 \cup \dots \cup S_k) \cap S_{k+1}) \geq \\ &\geq d(S_1 \cup S_2 \cup \dots \cup S_k) + d(S_{k+1}) \geq k+1, \end{aligned}$$

as required. ■

**Remark 2.12** *The sets  $S_1 = \{v_1, v_2\}, S_2 = \{v_2, v_3\}, S_3 = \{v_3, v_4\}$  are inclusion minimal independent sets of the graph  $H$  from Figure 3, such that*

$$d(S_i) > 0, S_i \not\subseteq \bigcup_{j=1, j \neq i}^3 S_j, i = 1, 2, 3.$$

*Notice that both families  $\{S_1, S_2\}, \{S_1, S_3\}$  have two elements, and  $d(S_1 \cup S_2) = 2$ , while  $d(S_1 \cup S_3) > 2$ .*

### 3 Conclusions

In this paper we investigate structural properties of  $\ker(G)$ .

Having in view Theorem 2.5, notice that the graph:

- $G_1$  from Figure 2 has only one inclusion minimal independent set  $S$  such that  $d(S) > 0$ , and  $d_c(G_1) = 1$ ;
- $G$  from Figure 3 has only two inclusion minimal independent sets  $S$  such that  $d(S) > 0$ , and  $d_c(G) = 2$ ;
- $H$  from Figure 3 has 6 inclusion minimal independent sets  $S$  such that  $d(S) > 0$ , and  $d_c(H) = 3$ .

These remarks motivate the following.

**Conjecture 3.1** *The number of inclusion minimal independent set  $S$  such that  $d(S) > 0$  is greater or equal to  $d_c(G)$ .*

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